Minimum “Energy” Embedding of a Tensegrity Structure

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Course: MAE 295 Networks and Control

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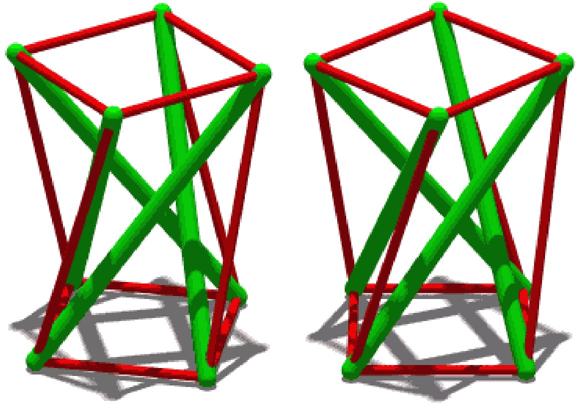
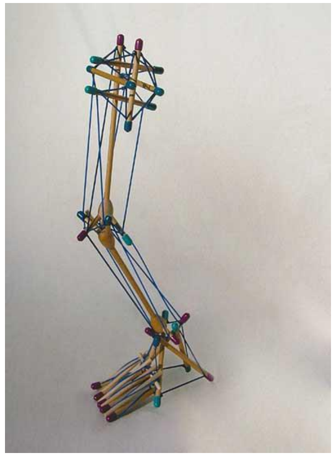
Background: Tensegrity Structures



T3-prism (Triplex)

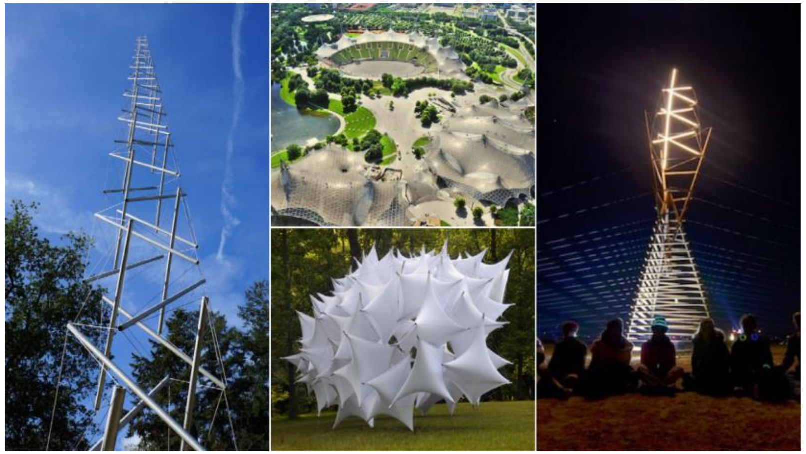
A system of isolated components under compression, such as rods (in green), inside a network of components under continuous tension, such as springs (in red).

Applications



Anatomy

Robotics

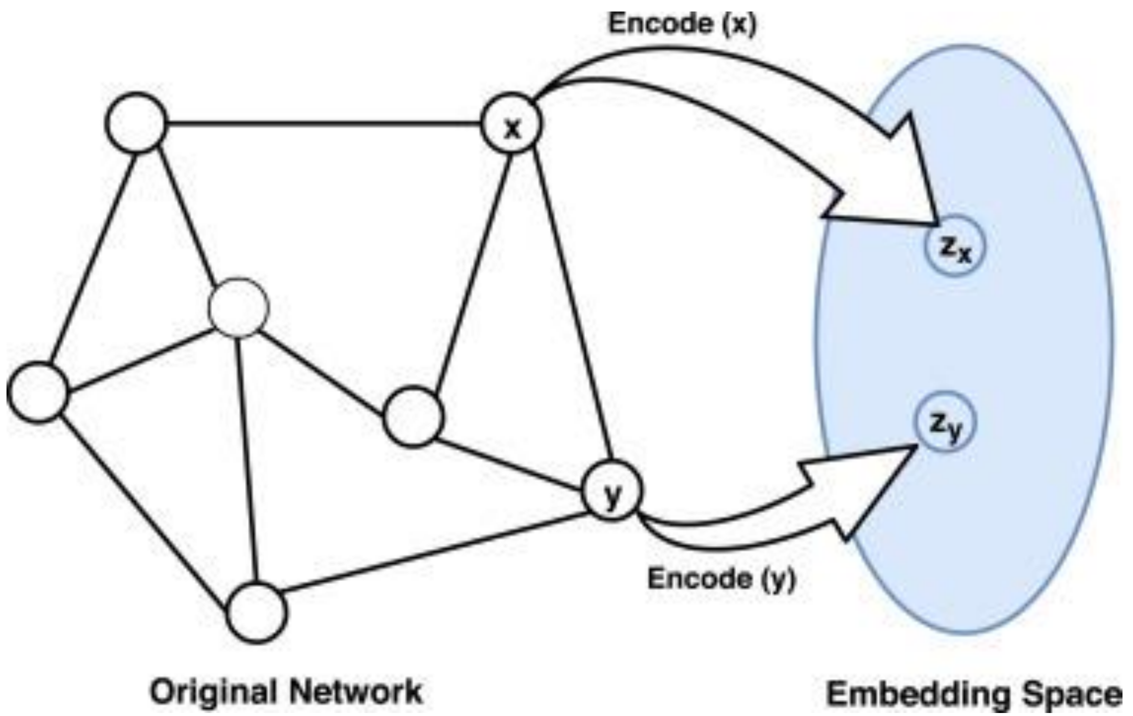


Architecture

https://www.magicalrobot.org/BeingHuman/2010/04/introduction-to-biotensegrity
https://www.nasa.gov/image-article/super-ball-bot/
https://www.arch2o.com/how-do-tensegrity-structures-defy-gravity-explained-with-10-examples/

Problem Setting

Consider the graph of a tensegrity structure $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Find an embedding in $\mathbb{R}^3$ that minimizes the "energy" stored in the springs.



Optimization Formulation

$$X^* = \underset{X}{\operatorname{argmin}} \ \operatorname{trace}(L_{\text{spring}} X X^\intercal)$$ "Energy" Stored in Springs

$$\text{s.t.} \quad \|x_i - x_j\|_2^2 = R_{ij}^2, \quad \forall i \in \mathcal{V}_{\text{rod}}, \forall j \in \mathcal{V}_{\text{rod}}$$ Fixed Distance Between Rigid Rods

$$\|x_i - x_j\|_2^2 \geq d_{\min}^2, \quad \forall i \neq j$$ Minimum Distance Between Between Nodes

$$\mathbf{1}^\intercal X = 0,$$ Graph Embedding is Centered at the Origin

where

$X \in \mathbb{R}^{n\times 3}$ is the 3-D position of each node;

$R_{ij} \in \mathbb{R}_{>0}$ is the fixed distance between each rigid rod;

$\mathcal{V}_{\text{rod}}$ is the index set of nodes for the rigid rods;

$d_{\text{min}} \in \mathbb{R}_{>0}$ is the minimum distance between any two nodes;

$\mathbf{1} \in \mathbb{R}^n$ is the identity vector.

Solved numerically using …

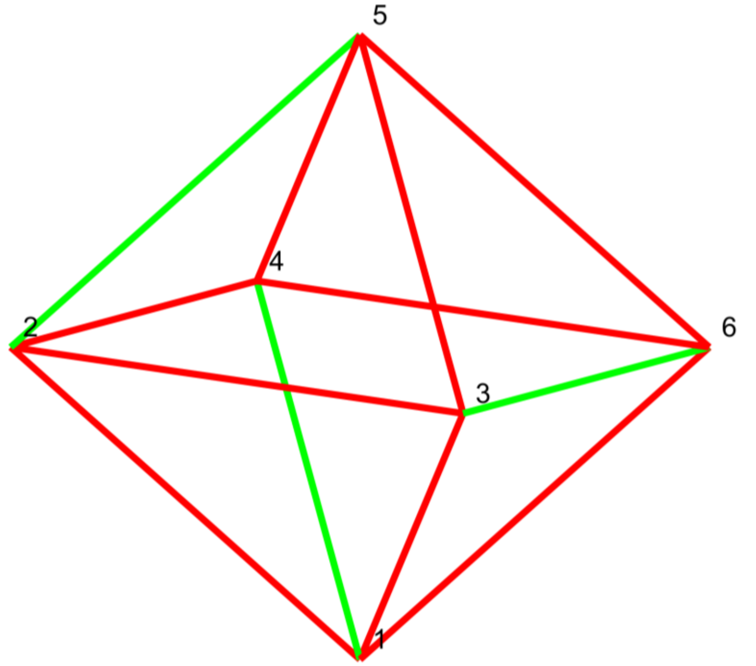
CasADi

J. A. E. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi: A software framework for nonlinear optimization and optimal control," *Math. Program. Comp.*, vol. 11, no. 1, pp. 1–36, Mar. 2019.

Numerical Example: Setup

$$A_{\text{rod}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_{\text{spring}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

where A_{rod} and A_{spring} are the adjacency matrix of the rigid rod and spring, respectively.



2D Graph (with Random Node Placement)

We assume that:

- $R_{14} = 2\ m$;
- $R_{25} = 2\ m$;
- $R_{36} = 2\ m$;
- $d_{\text{min}} = 1\ m$.

Initialization of X:

- Single realization results:

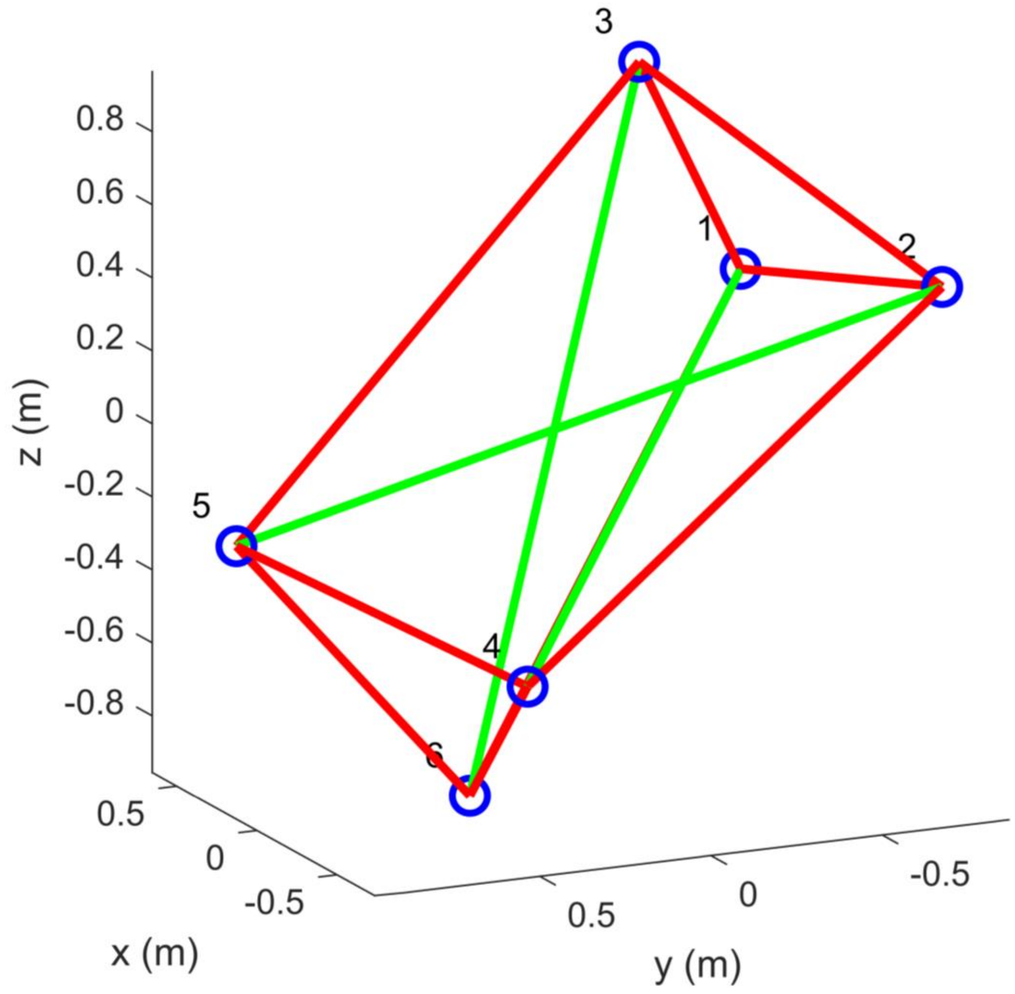
$$X_0 = \texttt{max}\{R_{14}, R_{25}, R_{36}\} * \texttt{ones}(6,3);$$

- 10,000 Monte Carlo (MC) trial results:

$$X_0 = \texttt{max}\{R_{14}, R_{25}, R_{36}\} * \texttt{rand}(6,3).$$

Numerical Example: Single Realization Results

(Locally) Optimal Solution



Graph Embedding of Nodes in $\mathbb{R}^3$:

$$X^* = \begin{bmatrix} 0.6532 & -0.7368 & 0.2296 \\ -0.3228 & -0.8698 & 0.4019 \\ 0.2015 & -0.2300 & 0.9638 \\ -0.7408 & 0.5354 & -0.4323 \\ 0.1254 & 0.9816 & -0.2074 \\ 0.0835 & 0.3196 & -0.9556 \end{bmatrix}$$

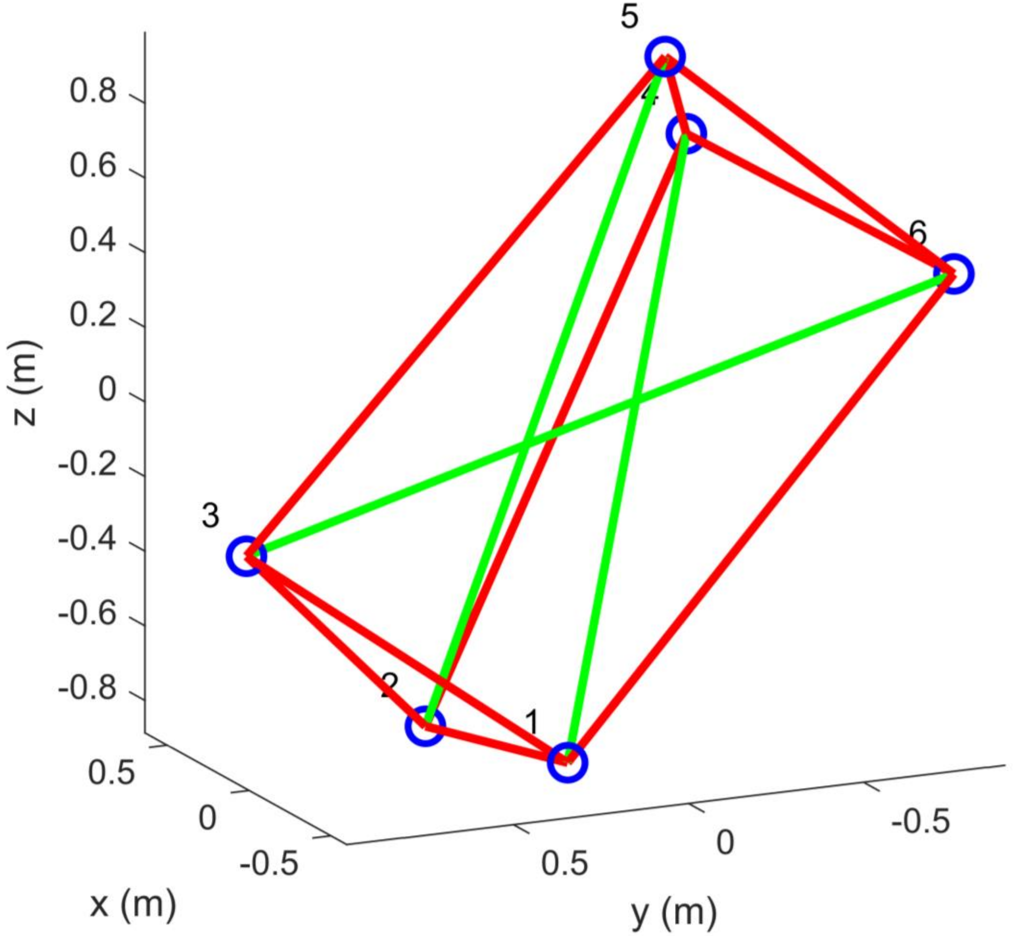
Energy Functional:

$$E^*_{\mathsf{spring}} = \mathrm{trace}(L_{\mathsf{spring}} X^* (X^*)^{\mathsf{T}}) = 14.54\ J$$

$$E^*_{\mathsf{rod}} = \mathrm{trace}(L_{\mathsf{rod}} X^* (X^*)^{\mathsf{T}}) = 12\ J$$

Numerical Example: Monte Carlo Trial Results

"Best" (Locally) Optimal Solution of 10,000 MC Trials



Graph Embedding of Nodes in $\mathbb{R}^3$:

$$X^* = \begin{bmatrix} -0.5983 & 0.3451 & -0.7384 \\ 0.3891 & 0.2907 & -0.8868 \\ 0.0143 & 0.9763 & -0.2627 \\ 0.6350 & -0.5688 & 0.5436 \\ -0.1514 & -0.1415 & 0.9896 \\ -0.2888 & -0.9018 & 0.3547 \end{bmatrix}$$

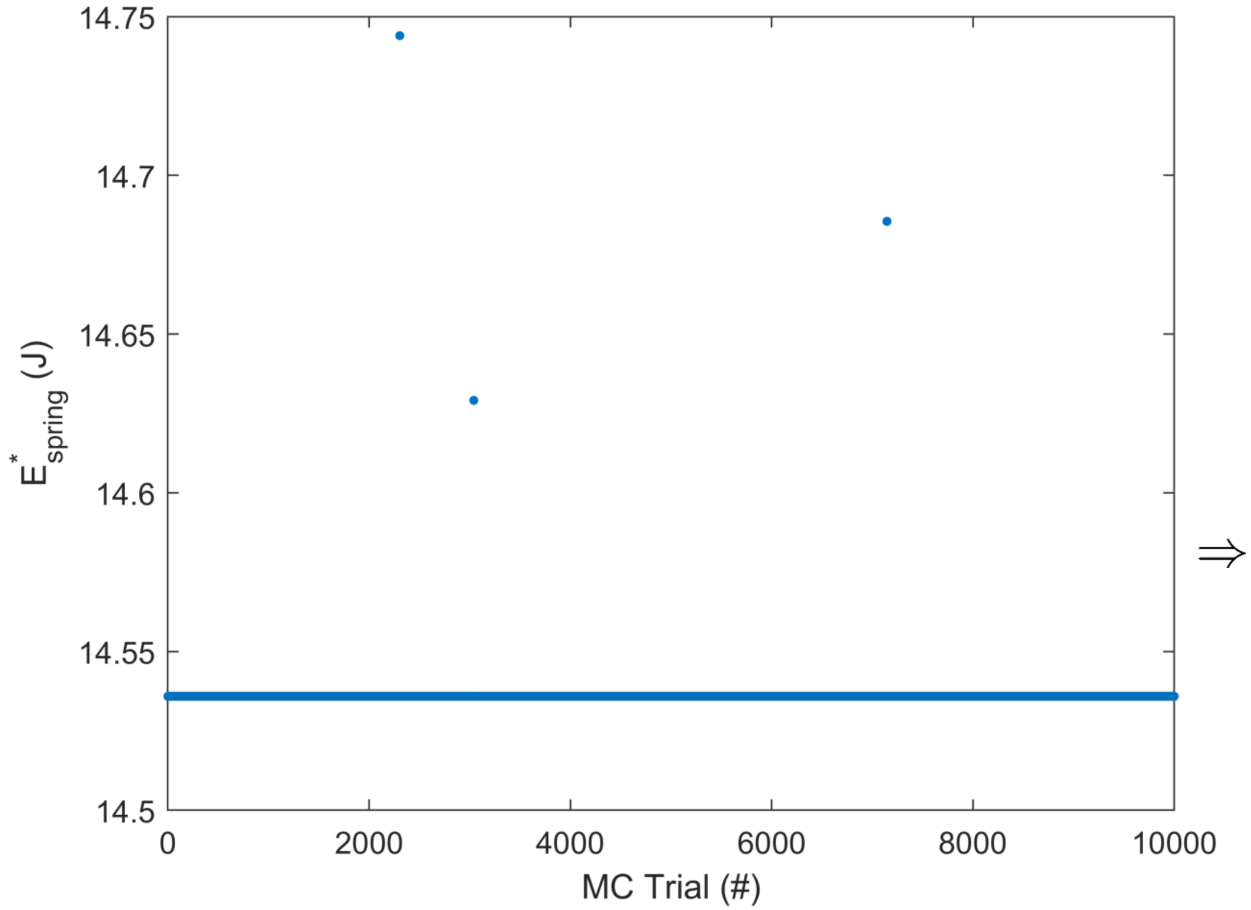
Energy Functional:

$$E^*_{\mathsf{spring}} = \mathrm{trace}(L_{\mathsf{spring}} X^* (X^*)^\mathsf{T}) = 14.54 \ J$$

$$E^*_{\mathsf{rod}} = \mathrm{trace}(L_{\mathsf{rod}} X^* (X^*)^\mathsf{T}) = 12 \ J$$

Numerical Results: Monte Carlo Trial Results

Energy Functional for (Locally) Optimal Solutions



Energy Functional:

$$E^*_{\text{spring}} = \text{trace}(L_{\mathsf{spring}} X^* (X^*)^{\mathsf{T}})$$

$\Rightarrow$ The optimal solution typically provides the lowest energy embedding (even with random initialization)!

Thank you for listening!